Four Color Theorem

by

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Report

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Master of Arts

The University of Texas at Austin

August 2009

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Dedication

I dedicate this report to my husband, who has always supported me, loved me, and had faith in me throughout the completion of my master's degree. And to my parents, who have always guided me down the right path.

Abstract

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The Four Color Theorem originated in 1850 and was not solved in its entirety until 1976. This report details the history of the proof for the Four Color Theorem and multiple contributions to the proof of the Four Color Theorem by several mathematicians. Ideas such as Kempe Chains, reducibility, unavoidable sets, the method of discharging, and the Petersen Graph are all covered in this report. There is also a brief discussion over the importance of a mathematical proof and how the definition of a proof has changed with the contributions of Computer Science to the mathematical community.

Table of Contents

List of Figures

Chapter 1: Introduction

The Four Color Theorem began as an intriguing problem in the early 1850s that led to a new field of mathematics called Graph Theory. The Four Color Theorem is of interest to teachers at the secondary level because of the varying levels and topics covered within the solution. Its intrigue is the basis of a heated controversy over the computer generated solution presented in 1976 by Appel and Haken, making mathematicians rethink what constitutes a complete proof.

From an educational standpoint, the Four Color Theorem can be introduced to students as early as primary school or at levels as high as doctoral work. Teachers of secondary level mathematics find the Four Color Theorem of interest due to the simplistic nature of the statement and the capability to be broken down and taught to students at any level. In 1886, the Headmaster of Clifton College posed the Four Color Theorem question to his school as a "challenge" for his students to work on, with the reward of having the student's work published in a magazine (Biggs, Lloyd and Wilson, 1976, p. 105). Still today, classroom teachers find the theorem just as intriguing as it was in 1886.

An overview of the problem is best stated by Gonthier (2008) where "the regions of any simple planar map can be colored with only four colors, in such a way that any two adjacent regions have different colors" (p. 1383). While the nature of this problem is simple, the solution is not. Since the conception of the problem by Guthrie in the early 1850s to the eventual solving by Appel and Haken in 1976, numerous mathematicians have tried solving this conundrum (Mitchem, 1981, p. 108). The Four Color Theorem's history and solution are so complex that the problem itself launched the study of chromatics and graph theory. It also fueled the philosophical debate of what consists of a computer "proof."

Chapter 2: History and Basic Terminology of the Four Color Theorem

The Four Color Theorem has intrigued scientists and mathematicians for over 150 years. It was not until 1976 that a "complete" proof was produced, aided by computer technology. The Four Color Theorem is often referred to as the Four Color Problem depending on the stage of completion, starting with its discovery by Francis Guthrie in the early 1850s. Guthrie developed the concept while studying the coloring of a map of England, realizing that it would take only four colors to distinguish the counties of England from one another (Mitchem, p. 108). After Guthrie discussed the theory of the Four Color Problem with his brother Fredrick, the Four Color Theorem was then passed around to multiple influential mathematicians in search of a proof. De Morgan, the inventor of mathematical induction, is officially credited with spreading the popularity of the Four Color Problem. (Biggs, Lloyd and Wilson, 1976, p. 92). Word of this problem spread throughout the mathematical world until it reached Pierce in 1860, which helped popularize the problem but the problem remained unsolved (Calude, 2001, p. 27). In 1879, Cayley published the first paper on the Four Color Problem entitled, "*On the Coloring of Maps*" which detailed why the proof of the Four Color Problem is so difficult to attain. In 1936, Konig published the first book on Graph Theory and in 1952 Kykin and Upensky published a book on graphing exercises (p. 27). All these works furthered the progress on the Four Color Theorem.

BASIC TERMINOLOGY

Before delving into the 'Who's Who' of the Four Color Theorem, it is essential to understand the important parts of the problem. Vocabulary and definitions must be presented so that each potential solver knows what can and can not be included in a solution of the problem.

Basic geometric ideas such as vertices, edges, and faces are integral pieces of understanding the Four Color Theorem; as well as, the basic concept of what constitutes a map or graph. Ringel and Youngs' (1968) definition of a *map* on a sphere infers no oceans can be considered in the coloring process and all countries must be connected. Countries or states being adjacent to each other may not be colored the same, but countries that only share a common point, or *vertex*, can be colored the same. Since an *edge* is defined as the boundary between two countries, it can also be said that no two edges can be colored the same (Ringel and Youngs, p. 438).

Maps in the context of coloring can be characterized by a *chromatic number*, χ. The chromatic number states the least amount of colors it takes to color the map. According to the Four Color Map Theorem, the chromatic number, m, should be

$$m \geq 4 . \qquad \text{(Ringel and Youngs, p. 438)}$$

The difficult part of proving the Four Color Theorem is the statement

$$m = 4,$$

since it is known that

$$m \leq 5$$

due to the Five Color Map Theorem (see Chapter 5: subtitle Heawood for the proof). To prove this statement, more information must be known about the problem in relation to the definition of a map and map vertices. A map's *vertices* can be discussed in terms of how many edges originate from each vertex called the *valence number* (Mitchem, 1981, p. 110). Malkevitch (2009) describes a *planar map* as a graph that can be drawn in a plane so that the edges meet only at the vertices where the edges do not overlap. Planar maps are important to this problem because a planar graph can be turned into a (*geometric) dual graph* when coloring maps (2009). To create a dual graph, countries are replaced by vertices and connected to other countries by an edge, forming a graph. To complete the dual graph, the vertices of the original graph are replaced by faces (Appel and Haken, 1989, p. 6). Malkevitch provides an excellent visual example of a dual graph, seen in Figure 1. Malkevitch states that this allows the mathematician to color the vertices of the graph instead of the country. The concept of dual graphs generalizes the map making it simple for mathematicians to label and manipulate it. From here, one can chose to color the vertices or the edges as an alternative means of proving the Four Color Theorem (2009).

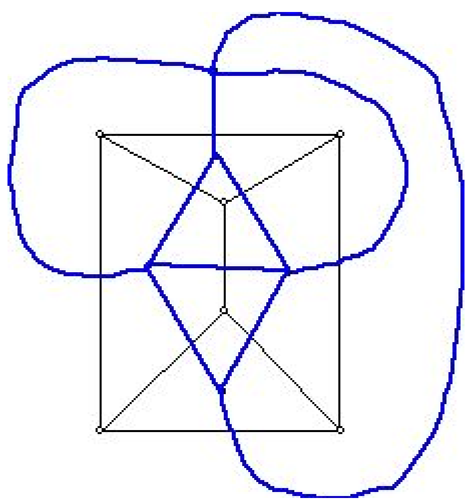

Figure 1: Creation of a dual graph from a planar graph. From "Colorful Mathematics: Part I" by Joseph Malkevitch, 2003, Retrieved March 03, 2009, from http://www.ams.org/featurecolumn/archive/coloring4.html

Chapter 3: Contributors and a Brief Overview of Their Proofs

KEMPE

Kempe (1879) has been the most famous fallacious solver of the Four Color Problem. For eleven years, Kempe basked in the glory of being the one to solve the Four Color Problem using his method called Kempe Chains. His work was so widely accepted he was made a Fellow of the Royal Society (Mitchem, 1981, p. 109) and knighted in 1912 (Calude, 2001, p. 28). Other mathematicians individually developed proofs during this time, but never to the magnitude or glory of Kempe. Ultimately, Kempe's work would be disproved by Heawood in 1890, then independently by De la Vallee – Poussin in 1896, and Errera in 1921 (Hutchinson and Wagon, 1998, p. 170).

HEAWOOD

Heawood's largest contribution is a slightly different proof named the Five Color Map Theorem (Biggs, Wilson and Lloyd, 1986, p. 105). It is worth mentioning that Kempe's work is the backbone of the Five Color Theorem (Calude, p. 28). Other contributions from Heawood include the Heawood Inequality and the Heawood Map-Coloring Conjecture. These ideas will be developed later in Chapter 4.

ERRERA

Errera's contribution to the Four Color Problem was the development of a new method of coloring the map, even though it did not solve the problem. His idea of interchanging colors on a map was similar to Kempe Chains. In fact, Kempe Chains work in approximately ninety percent of random labelings of Errera's work (Calude, p. 28).

TAIT

Subsequently, while Kempe's work was being disproved by Heawood and Errera, Tait (1880) developed a new way to solve the problem by means of a linear graph. Most studies to this point have concluded that for the Four Color Theorem to be proven, three boundary lines must converge at each vertex, defined as a *trivalent vertex*. Tait's contribution to the Four Color Theorem was to color the edges of the graph instead of the regions within the map (Biggs, Wilson and Lloyd, 1986, p. 103).

PETERSEN

Until this point in the history of the Four Color Theorem, most mathematicians have reused works of Kempe, Heawood, Tait, and Hamilton to create differing versions of the Four Color proof. One of the most important fallacious proofs following Tait comes from Petersen, designer of the Petersen Graph and the Petersen Theorem. These proofs are developed, in detail, later in this report (Wilson, 2002, p. 139).

BIRKHOFF

Following Petersen, Birkhoff's work focused on the concept of configuration reducibility and a special configuration he called Birkhoff's Diamond. *Birkhoff's Number* became another major contribution to the solving of the Four Color Theorem by finding the maximum number of regions within a map that can still be four colorable starting with twenty two regions. After Birkhoff's discovery, the number of four colored regions increased significantly starting with Franklin. Franklin (1922) used Birkhoff's methods in combination with Kempe Chains and nonreducible regions to produce a map containing twenty five or fewer regions that can be colored in four colors (p. 235). Franklin's method uses polygons where n, the number of sides, was greater than or equal to five, as well as, the concept of *unavoidable sets*. Reynolds (1926) proved 28 regions, followed by Franklin again with 32 regions in 1938, then Winn (1940) with 36 regions, and finally, Ore and Stemple (1970) with 40 proved regions. Just prior to Appel and Haken solving the problem, the count was increased to 90 regions in 1976 by Mayer (Birkhoff, 1913, p. 116).

HEESCH

In 1950, Heesch was the first mathematician after Kempe to believe that the Four Color Theorem could only be proven by finding an unavoidable set of reducible configurations. The problem with this train of thought was that to find a set (map) that could be restricted to known configurations, the number of configurations would approximate ten thousand (Appel and Haken, 1989, p. 6). Heesch, aided by technology, started a proof of the Four Color Theorem that would encapsulate such a large number of configurations. Heesch's ideas on how to solve the Four Color Theorem led to the development of discharging (see Chapter 5: subtitle Method of Discharging for further explanation). Discharging was an integral concept of Appel and Haken's proof of the Four Color Theorem (Appel and Haken, 1976, p. 711).

APPEL AND HAKEN

Appel and Haken successfully solved the Four Color Theorem in 1976, but met much pessimism from the mathematical community because their proof was mostly derived using computer technology. Even though it was widely challenged, Appel and Haken are the first mathematicians to prove the Four Color Theorem without major error (Calude, 2001, p. 2).

Chapter 4: The Mathematics Behind the Four Color Theorem

The Four Color Theorem has several conditions that must be met. Appel and Haken's (1976) definition of the Four Color Problem states that "Every planar map can be colored with at most four colors" (p. 711). Kempe (1879), states that the Four Color Problem also includes the following definition:

> A practical way of coloring any map is this. Number the districts in succession, always numbering a district which is less than six boundaries, not including those boundaries which have a district already numbered on the other side of them. When the whole map is numbered, beginning with the highest number, letter the districts in succession with four letters, *a, b, c, d,* rearranging the letters whenever a district has four round it, so that it may have only three, leaving one to letter the district with. When the whole map is lettered, color the districts, using different colors for districts lettered differently (Hutchinson and Wagon, 1998, p. 170).

Each mathematician's work has led to a more detailed proof from the next proposed solver. Even Appel and Haken's solution to the Four Color Theorem involves work developed by the earliest solvers of the problem.

KEMPE

Kempe's proof produced two important ideas. The first was that all vertices of a map must be at least of degree five (Appel and Haken, 1989, p. 711). Kempe's second idea was a theory called *Kempe Chains*, which is widely used throughout the mathematical community to this day and has been thought of as the foundation of the Four Color Theorem. Brahana (1923) describes Kempe Chains as a method of rearranging the colors on a map by switching adjacent colors. Kempe Chains are created in the form of an *AC* chain and a *BD* chain. Chains can not cross each other, leaving the manipulator with a choice to either use an *AC* chain or a *BD* chain. Using Figure 2 to model Kempe Chains, an *AC* chain can be utilized by switching the *C* to an *A*, leaving room for the middle block to become a *C* and proving the Four Color Problem. Likewise, the same happens if the manipulator chose to change *D* with *B*, leaving the middle block to become color *D* (p. 237).

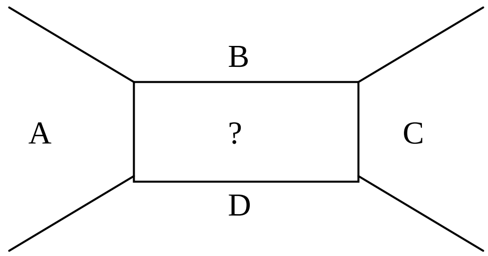

Figure 2: Visual Example of Kempe Chains. From "Colorful Mathematics: Part I" by Joseph Malkevitch, 2003, Retrieved March 03, 2009, from http://www.ams.org/featurecolumn/archive/coloring1.html

Kempe hoped to prove the Four Color Theorem by stating that if a simple map could be colored with four colors, a more complex map could be reduced to a simpler map and it too can be colored with only four colors. Since every map that is more complex could be broken into smaller simpler maps colored in a manner of four colors, all maps can be four colorable (Biggs, Wilson and Lloyd, p. 107).

HEAWOOD

Heawood's two contributions to the Four Color Problem are the Heawood Inequality and the Heawood Map-Coloring Conjecture. To understand Heawood's Inequality, a background on Euler's formula is necessary. Euler's Formula states:

$$N_v - N_e + N_f = 2. \tag{1}$$

This can be proven through deduction by looking at a few examples where the number of vertices, N_v, minus the number of edges, N_e, plus the number of faces, N_f, always equals two.

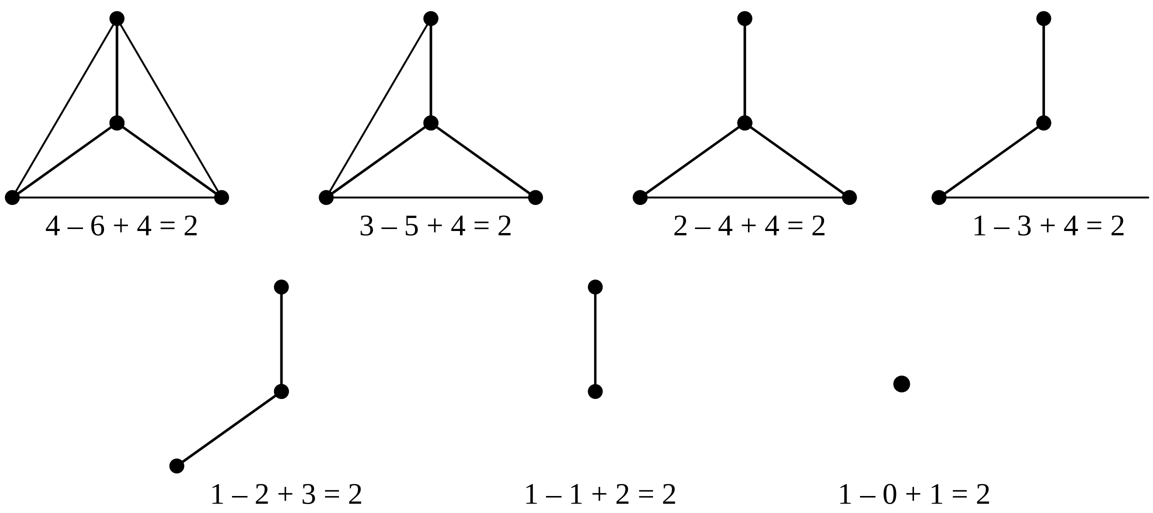

Figure 3: Proof of Euler's formula. From "Four Colors Suffice" by Robin Wilson, 2002, Princeton University Press, New Jersey, p. 51.

By looking at each figure in succession, in Figure 3, it can be seen that by reducing a hanging edge from each map, Euler's Formula equals two. When applying this to the Four Color Theorem, Heawood wanted to consider a map with *F* countries, *E* boundaries and *V* meeting places. Figure 4 is a basic map that fits this description.

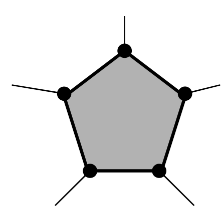

Figure 4: Pentagon with three edges at each vertex. From "Four Colors Suffice" by Robin Wilson, 2002, Princeton University Press, New Jersey, p. 51.

Since each vertex must be trivalent, *3V*, and each edge counts the number of vertices as well, the vertices are double counted. Heawood accounted for this "double counting" by taking the trivalent vertex and dividing by two to create the following statement:

$$2N_e \geq 3N_v\,. \tag{2}$$

Additional statements can be made by comparing the fact that every edge is bounded by two faces

$$\alpha N_f = 2N_e\,, \tag{3}$$

where α is the average number of edges surrounding a face (Biggs, Lloyd, and Wilson, p. 110). Heawood proves the fact that

$$m \leq 5$$

(stated in Chapter 2) by contradiction, by stating that the problem will assume six countries instead of five surrounding the main region on a map. Therefore, six boundaries are produced, $6N_e$. Similarly, by the same argument of double counting vertices, boundaries have also been double counted when comparing the faces and edges, giving the inequality

$$N_e \geq 3N_f\,. \tag{4}$$

Combining equations (1), (2), and (4), the problem yields

$$N_f - N_e + N_v \leq \frac{1}{3}N_e - N_e + \frac{2}{3}N_e = 0\,. \qquad \text{(p. 54)}$$

Through contradiction, we know that a region surrounded by six countries must equal two as stated by Euler's Formula in equation (1). Thus, the region must have five or less colors to be colorable.

Moving on to the Heawood Inequality, the same statements can be used with a slight modification. Equation (1) will now be known as the *Euler characteristic* and will equal

$$N_v - N_e + N_f = \chi(S_p)\,. \qquad \text{(Biggs, Lloyd, and Wilson, p. 110)}$$

The object of the Heawood Inequality is to prove that the chromatic number of a surface with p holes, or $\chi(S_p)$, can be satisfied if p (genus) is positive. Heawood was able to prove, using the same form of argument as above, that the chromatic number, $\chi(S_p)$,, satisfies the inequality

$$\chi(S_p) \le \left[\frac{7+\sqrt{1+48p}}{2}\right], \text{ if } p > 0.$$

Concurrently, Heawood was able to produce a torus with seven countries on it, where each country was adjacent to all the others. Since negative values can be excluded (no negative holes) and only integers can be used (can not have half a hole), Heawood concluded that

$$\chi(S_1) \le 7\ .$$

This would mean that a surface with genus of one, like a *torus* or three-dimensional donut shape, must have a chromatic number less than or equal to seven. This inequality would be known as the Heawood Map-Coloring Conjecture. Only a year later, another mathematician, Heffter would prove that the inequality also proves true for values of

$$1 \le p \le 6$$

and a few additional values after six. Heawood's and Heffter's conjectures are still in good standing within the mathematical community today (Ringel and Youngs, 1968, p. 439).

ERRERA

While Heawood's disproof of Kempe's work led to the Five Color Theorem, Errera's disproof is a counterexample of Kempe Chains. Errera's counterexample shows a better visual illustration of Kempe's oversight than the construction of Heawood's methods. Figure 5 is a graph labeled to Kempe's specifications. Errera then uses this arrangement to color each vertex as Kempe did. Errera makes use of Kempe Chains until the top of the vertex is reached, which coincidentally is the only uncolored vertex on the whole map. This is where Kempe Chains fail. No color can be placed in the last vertex once the rest of the map has been colored as in Figure 6.

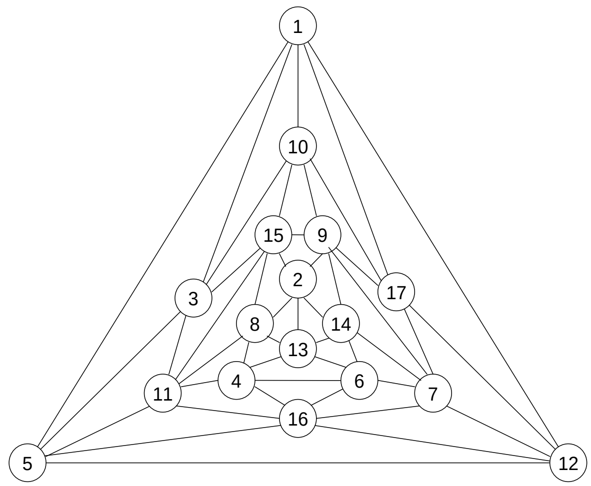

Figure 5: An example of Errera's set up for the counterexample of Kempe's proof. From "Kempe Revisited" by Joan Hutchinson and Stan Wagon, 1998, *Mathematical Association of America*, *105*(2), p. 170.

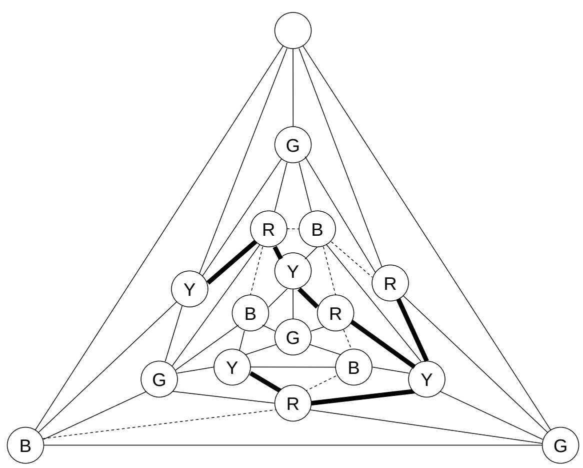

Figure 6: The counterexample of Kempe's proof. From "Kempe Revisited" by Joan Hutchinson and Stan Wagon, 1998, *Mathematical Associatioin of America*, *105*(2), p. 170.

In further detail, Figure 6 shows how Kempe Chains are used for the red-yellow chains (dark line) and red-blue chains (dashed lines). Since only the top vertex needs to be colored, no attempt of Kempe's Chains will allow vertex one (Figure 5) to be a color already used. The vertices colored green at five and twelve can not be eliminated or a fifth color would have to be used on vertex one (Hutchinson and Wagon, p. 170).

Errera wrote a paper entitled *du Coloriage des Cartes et de quelques Questions d'Analysis Situs*, with the hope of completing the proof of the Four Color Theorem. Sadly, Errera disproved himself while finishing the paper, never actually developing a complete proof. Errera believed in Kempe Chains and thought they were the backbone of solving the problem. Errera's paper presented a method of interchanging colors until a region(s) is missing, then allow that region to become another color. Errera found that after twenty operations of color changes, the colors occupy the same positions as in the beginning. Errera's work concluded by saying that the whole map must take less than twenty colorings, but by looking at color changes in multiples of twenty, one can return the map to its original state (Hutchinson and Wagon, p. 170).

TAIT

Tait hoped to show that three colors appear at every vertex as shown in Figure 7 (p. 103).

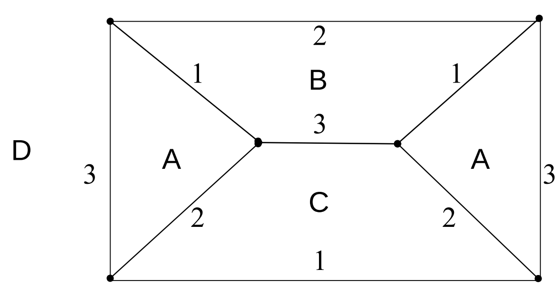

Figure 7: Tait's proof where each vertex must have a meeting of three colors. From "Graph Theory, 1736 – 1936" by N.L. Biggs, R.J. Wilson, and E.K. Lloyd, 1986, Oxford University Press, Oxford, p. 103.

The colors, *A*, *B*, *C*, and *D* in Figure 7, are four different colors on a map, while the numbers 1, 2, and 3 are the colors of the edge. Using the scheme below, Tait labeled the edges to show that each vertex has three colors associated to it, leaving the fourth color for the vertex.

$$1 = \frac{A}{B} = \frac{C}{D}$$

$$2 = \frac{A}{C} = \frac{B}{D}$$

$$3 = \frac{A}{D} = \frac{B}{C}. \qquad \text{(Brahana, 1923, p. 239)}$$

Tait explains that the converse works as well. If the numbers are already on the edges of the graph, then the colors can be assigned. Tait believed Kempe was right and that his elementary theorem by induction would convince the world that the Four Color Theorem was proven. Tait's proof turned out to be yet another fallacious attempt at the problem (Brahana, p. 239). Tait's theory is false because Tait assumed that every three-connected planar graph incorporates a *Hamiltonian Circuit*. Coxeter (1971) explains that a Hamiltonian Circuit is where a vertex is replaced by a polygon so that each vertex in the new graph can be 'visited' just once from start to finish if the graph were a path. If this were true, Tait's proof would be correct; however, Tutte disproved Tait through a counterexample where a graph was given with no Hamiltonian Circuit making Tait's argument for having a Hamiltonian Circuit a *sufficient*, but not *necessary* condition (p. 276).

PETERSEN

Petersen explored the idea of factoring graphs of even and odd degree. While even degree graphs can be decomposed to graphs of degree one and two, odd degree graphs can not be decomposed to any degree (Biggs, Lloyd and Wilson, p. 197). Petersen devoted the majority of this time to the study of odd degree trivalent graphs which is the main composition of Petersen's Theorem. The underlying concept in Petersen's proof is the work constructed by Tait. While using Tait's work, Petersen discovers a graph that does not apply to Taits theories. The rationale in disproving Tait can be seen in the Figure 8.

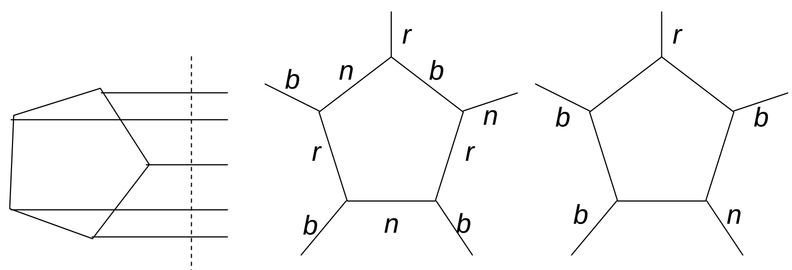

Figure 8: Set up and counterproof to Tait's argument. From "Graph Theory, 1736 – 1936" by N.L. Biggs, R.J. Wilson, and E.K. Lloyd, 1986, Oxford University Press, Oxford, p. 103.

Petersen uses the left drawing in Figure 8 to show that a pentagon will have five edges attached to each vertex, making each vertex trivalent. Using Tait's proof, two examples can be seen. Since there must be three of one color and one each of the two remaining colors, *b*, or blue has been chosen to repeat three times on the new edges. While the middle drawing in Figure 8 is a correct example of Tait's proof, the drawing on the right in Figure 8 is a counterexample to Tait's logic (Biggs, Wilson and Lloyd, p. 198).

Petersen is most famous for the Petersen Graph which is the ultimate counterexample to Tait's work, which shows no Hamiltonian cycle within the constructed polyhedra. Several different transformations of the Petersen Graph can be found within the literature on the Four Color Theorem. Wilson (2002) says that Petersen derived the Petersen Graph from Kempe, who created another transformation of the graph twelve years prior to Petersen's publication in 1898 (p. 139). These transformations can be seen in Figure 9.

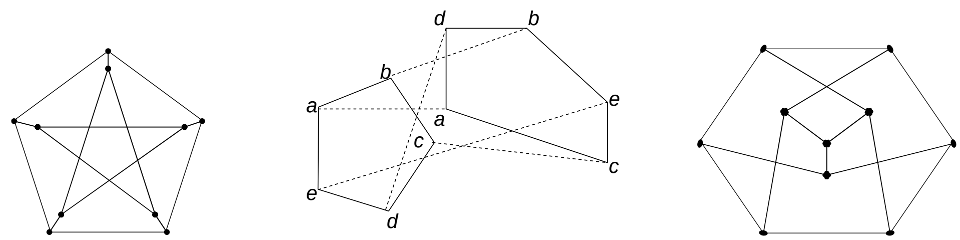

Figure 9: Transformations of the Petersen Graph from left to right: usual form, Petersen's Graph, Kempe's early version. From "Four Colors Suffice" by Robin Wilson, 2002, Princeton University Press, New Jersey, p. 139.

Moving on to Petersen's Theorem, one can see it is comprised of four sub-theorems as stated below.

> Theorem I. Given a simple graph of order greater than two, we can always obtain from it a simple graph of lower order by removing any 1-cell and properly joining the four incident 1-cells in pairs.
> Theorem II. Every cell of a colored simple graph is on a closed red-blue path and hence can have its color changed.
> Theorem III. Every simple graph is colorable.
> Theorem IV (Petersen's Theorem). A regular graph of the third degree with fewer than three leaves is colorable. (Frink, 1926, p. 491)

Petersen's Theorem builds on a cubic graph by redefining the definition of primitive. Frink (1926) states that a *primitive* graph is a graph that can not be obtained by connecting two regular graphs of a lower degree. Petersen's Theorem also states that every third degree primitive graph must have three or more *leaves*, or connected portions, joined by exactly one line/edge, which Frink calls *1-cells*. By using Figure 10, it is shown how Petersen intended to have the graph (a second order cubic) broken into sub-graphs by deleting the 1-cell, *x*. This is the definition of *factorable,* which makes each of the stems factors of second degree. Each disconnected factor must have a path to the original graph. If a path does not exist from the stem to the 1-cell, then it is then called an *isthmus*. The primary reason for splitting the cells into smaller reducible cells (specifically decomposed to a first and second order factor) is to keep the circuit intact, proving Theorem I (p. 491).

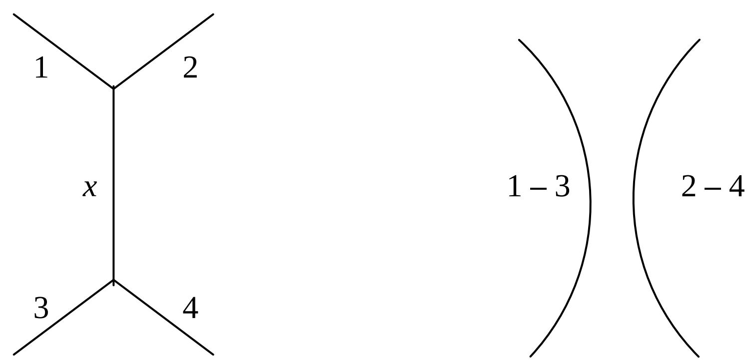

Figure 10: Petersen's proof of removing *x* in the graph on the left to obtain the graph on the right, attaining a simple graph. From "A Proof of Petersen's Theorem," by Orrin Frink Jr., 1926, *The Annals of Mathematics, Second Series, 27*(4), p. 491.

After showing factorization of the graph, Frink states first degree 1-cells should be colored red and second degree 1-cells should be colored blue. This leads to the definition of *colored*, meaning every vertex has two blue 1-cells and one red 1-cell. Frink proves Theorem II through contradiction. Frink used Kempe Chains and stated that a red-blue pattern on a closed path can be interchangeable.

Three cases are used to prove Theorem III by using a simple second ordered graph. Case 1 states that after factoring, the two new 1-cells should be colored blue and with the addition of the deleted red cell the original case exists. Case 2 is the same as Case 1 except the two new cells are colored red and blue respectively, attaining the original graph with the addition of the deleted 1-cell. Case 3 utilizes Kempe Chains and Theorem II, stating that if the two new 1-cells are red, then a color must be changed to attain the new graph (Frink, 1926, p. 492).

Petersen's Theorem proves a simple graph is colorable, but a graph containing only one leaf is impossible. The Theorem further states that if two leaves are given a color, one can add a new 1-cell to create a properly colored graph and a perfect image of the original (Frink, 1926, p. 493).

Chapter 5: The Solution

There are really only two main parts to proving the Four Color Theorem. Mitchem's (1981) proposal to solve the problem is to find an unavoidable set, *S*, and show that each element of *S* is reducible. Once these two conditions have been met, the theorem is solved. The object is to have the smallest reducible graph that can not be contained within another graph (Mitchem, 1981, p. 114).

UNAVOIDABLE

Throughout Kempe's proof, it was shown that no normal map can contain countries with six or more borders making the Four Color Problem limited to maps with countries of five or less borders. If one country in every map is considered to be bordered by two, three, four or five other countries, then those countries are considered *configurations* of the map. *Unavoidable* means that every map must be made of countries surrounded by only two, three, four or five other countries (Appel and Haken, 1989, p. 4). Furthermore, *unavoidable sets* are of degree 5 and must contain one member of the set of configurations. Malkevitch (2009) describes a *degree* as being the number of edges originating from each vertex. In this case, a vertex with five originating edges is said to be of *degree 5* or *5-valent*. To produce unavoidable sets, the method of discharging must be used. This method and a further explanation of the solving of this problem will be discussed in further context when talking about the solution from Appel and Haken (p.711).

REDUCIBLE

Kempe also was the first mathematician to introduce the concept of reducibility. The concept of *reducibility* comes from the ability to look at a map and examine the configurations and chains within the map and proving the map cannot be a minimal five chromatic map (Appel and Haken, 1989, p. 5). Upon a closer examination of Kempe's work with Kempe Chains, it was discovered that if a simple five chromatic map is given, containing a country with four colors, it can always be manipulated to be four colorable. Appel and Haken (1976) considered a configuration to be reducible if every part of the graph could not be disproven with a proven counterexample using known configurations. After each part of the map has been reduced, then one must prove it is unavoidable and vice versa (p. 712).

BIRKHOFF'S REDUCIBLE ARRANGEMENTS

Prior to Birkhoff's work on reducible arrangements in 1913, only three successful reducible facts were known about the solving of the Four Color Theorem. The first fact is that if

$$v > 3,$$

where v is a vertex, then the coloring of the associated map can be reduced to fewer regions. Secondly, "if any region of a map is *multiply-connected*, the coloring of the map may be reduced to the coloring of maps of fewer regions" (Birkhoff, 1913, p. 115). This means a map can be broken into partial maps and colored. When all the partial maps are reconciled, the map proper will maintain the same colors as its partials. Lastly, "if the map contains any 1-, 2-, 3-, 4-sided region, the coloring of the map may be reduced to the coloring of a map of fewer regions" (p. 115). This concept can be accomplished by shrinking a region to a point, coloring the remaining map in three colors, then reintroducing the point as a different colored region, making the map four colorable.

Birkhoff's (1912) contribution was the first reducible arrangement known as Birkhoff's Diamond and is shown as Figure 11.

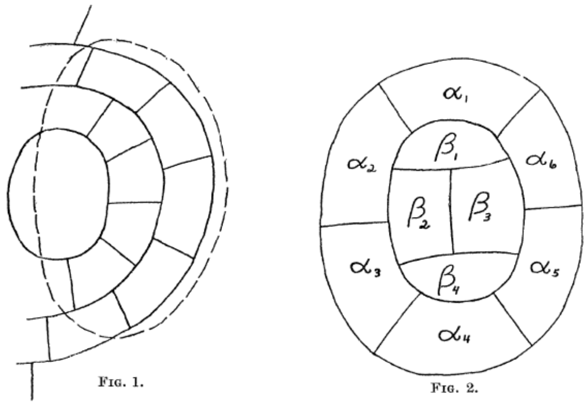

Figure 11: An example of Birkhoff's Diamond. From "The Reducibility of Maps," by George D. Birkhoff, 1913, *American Journal of Mathmatics, 35*(2), p. 125.

Birkhoff (1913) explained his proof by labeling the inside region $M_{1,}$ comprised of regions β_1, β_2, β_3, β_4; and the outside six regions, ring *R*, comprised of α_1 through α_6. Birkhoff showed, through contradiction, that *R* was not reducible with respect to M_1. Birkhoff's proof showed in a completely reduced map, no boundary line can be enclosed by four 5-sided regions. Furthermore, once a reducible set had been defined, Birkhoff established the maximum number of four-colorable regions within a map was twenty-two regions, otherwise known as Birkhoff's Number (see Chapter 3: subtitle Birkhoff) (Birkhoff, 1913, p. 125).

METHOD OF DISCHARGING

Once the concept of reducibility and unavoidable sets had been established within the Four Color Theorem, it was only matter of time before the missing link would be found. The method of discharging is that missing link. Appel and Haken (1976) explain the process prior to discharging by labeling the graph until the charge of the whole graph is positive. Each vertex with degree *k*, underwent a labeling process where each vertex was given a charge using the algorithm 6 – *k* (Wilson, 2002, p. 190). This process signifies all vertices of degree five or less will have positive charges and all vertices of degree six or greater will receive negative charges (Appel and Haken, 1976, p. 711). Mitchem (1971) gives a good explanation of Heesch's method in a slightly different format. *Discharging* is described by giving every vertex in a map a charge, such as *6-degv*, where the charge is six and the vertex being described is vertex *v*. Euler's Theorem states that all charges of a graph must sum to twelve, so the discharging procedure moves charges around on the graph, while keeping the sum at twelve. The amount of decrease from one charge must be equal to the amount of increase to another charge. Since the charges are being distributed, the sum of twelve is never changed, rather redistributed over the graph (Appel and Haken, 1989, p. 712). The name discharging comes from when an *n*-degree vertex loses its positive charge becoming "discharged." Concurrently, if a major vertex gains so much charge that it becomes positive, then it is deemed "overcharged" (Appel and Haken, 1989, p. 7).

The discharging procedure is a counterexample to each case addressed in the Four Color Theorem. The object is to make all the vertices gain negative charges and have the overall sum for the graph not equal to twelve. This proves two things. The discharging procedure proves that no element of set *S* can be in the graph and that *S* is unavoidable (Mitchem, 1981, p. 115). Appel and Haken (1989) describe the process as a "precise procedure" of moving charges around so that all vertices have a positive charge and the "resulting distribution must be in a reducible configuration." These conditions create an unavoidable set, thus proving the Four Color Theorem (p. 7). Using principles from conservation of charge, an algorithm was developed by Koch in conjunction with the work of Appel and Haken to solve the Four Color Problem (Appel and Haken, 1976, p. 712).

SOLUTION

Appel and Haken's (1976) solution to the Four Color Problem was an algorithm that used reducible sets of ring size fourteen or smaller that consisted of less than two thousand configurations. Since the majority of the work was done using the speed and repetition of a computer, many of the techniques and discharging procedures were checked by hand prior to publishing by Appel and Haken's own children. To this day, no solution to the Four Color Theorem exists without the use of computer technology. Appel and Haken are the first, but not the last mathematicians to correctly solve the Four Color Theorem. Work of other mathematicians after Appel and Haken have reduced the number of configurations needed to prove the Four Color Theorem or have tweaked the method of discharge to be more concise. But no mathematician has been able to disprove the proof generated by Appel and Haken thirty three years ago (Appel and Haken, 1976, p. 712).

Chapter 6: Conclusion

The solving of the Four Color Problem by Appel, Haken and Koch was a huge milestone in the history of the Four Color Theorem. Some skeptics still believe the Four Color Theorem does not have a solution since it was solved by a computer. The reducibility of configurations by a computer has been studied and analyzed extensively by Heesch, Gill, Allaire and Swart. Koch's computer programs and algorithms were not designed to replace mathematics, but to use the charging procedures in a more flexible and efficient manner (Appel and Haken, p. 712). Allaire (1977) produced a proof similar to that of Appel and Hakens and in 1996. Concurrently, Robertson, Sanders, Seymour and Thomas developed a more powerful computer program that decreased the amount of time to solve the problem and increased the efficiency of the discharging methods. Appel and Haken's proof consisted of fifteen hundred cases while the 1996 proof only needed six hundred thirty three cases for the mathematicians to be confident in their result (Calude, 2001, p. 28). To this day, Appel and Haken's proof of the Four Color Theorem still stands with minor corrections.

COMPUTER GENERATED PROOFS

It is worth mentioning the controversy caused by the Four Color Theorem. The Four Color Theorem is the first partial computer generated proof that has not been solved without the use of a computer. The proof has caused much speculation as to what a proof is and what constitutes a properly done proof (Mitchem, 1981, p. 116). This has opened the door for professional discussions on what a proof is and how proofs can be incorporated into modern technological mathematics through numerous articles over what a proof is, the history of proofs, and computer aided proofs. Many mathematicians, such as Halmos and Hersch, argue that proofs should cause enlightenment to the solver and should be a learning tool to discover the methods needed to solve problems and not solely to find answers (Calude, 2001, p. 31). While this argument still rages, it can be noted that computers are more a part of the world than ever and without them the Four Color Theorem would still be the Four Color Problem.

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Vita

Kimberly Ann Calton was born in Phenix City, Alabama on May 11, 1981, the daughter of Deborah Ann Pienkowski and John Pienkowski. After completing her work at East Coweta High School, Sharpsburg, Georgia, in 1999, she entered the United States Military Academy in West Point, New York. Completing a Bachelor's of Science Degree in Environmental Science in 2003, she proudly served the United States Army. She met her husband at Engineer Officer Basic Course (EOBC) and was married on July 10, 2004 at Fort Hood. She served as a Platoon Leader, Executive Officer and Assistant S-4 of the 62nd Combat Engineer Battalion, Fort Hood, Texas following graduation from EOBC. In 2006, she was honorably discharged from the United States Army with the following awards: Army Commendation Medal, Army Achievement Medal, National Defense Service Medal, Global War on Terrorism Service Medal, and Army Service Ribbon. During the following years, she was employed as a Mathematics Teacher at Tippit Middle School in Georgetown, Texas; a Mathematics Teacher and Department Head at Manor High School, Manor, Texas; and a Mathematics Teacher at Brownwood High School, Brownwood, Texas, where she currently is employed. In June 2007, she entered the Graduate School at the University of Texas at Austin where she is a member of Phi Kappa Phi, a collegiate honor society.

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